ON SOME GENERALISATIONS OF THE GOLDEN PROPORTION AND OF THE GOLDEN RECTANGLE

Cristina-Elena Hrețcanu*, Adara-Monica Blaga, Ciprian-Ionel Hrețcanu*****

*Stefan cel Mare University of Suceava, Faculty of Food Engineering, 13 Universitatii St., 720229, Suceava, Romania e-mail: cristinah@usv.ro
**Facultatea de Matematică, Universitatea de Vest din Timişoara, e-mail: adara@math.uvt.ro
***Colegiul de Artă "Ciprian Porumbescu" Suceava

Abstract: *The purpose of the present paper is to define and study some properties of the generalized Golden proportions which arise as roots of the equation $x^{p+1} - mx^p - n = 0$ (m,n,p are natural numbers) named by us generalized Golden (p,m,n)-proportions. The concept of the golden (p,m,n) – proportions extends a number of new mathematical constants, which can be useful in finding new applications. Also, we give a generalization of the golden rectangle, named by us "metallic" rectangle, which is a rectangle based on the metallic numbers introduced by Vera Spinadel.*

Keywords: *Golden proportion, Golden rectangle, metallic rectangle, generalized Golden (p,m,n)-proportion*

Introduction

The Golden proportion (also called as the Golden section, Golden mean, Divine ratio, Divine proportion, sacred cut) is a ratio found in fundamental forms: plants, flowers, viruses, DNA, shells, planets and galaxies.
Throughout the history of art and architecture, traditional artists adopted the Divine Ratio as sacred measure and aesthetic proportion in order to embody the spirit in the matter. The Great Pyramid in Egypt contains this ratio [Kocer, 2007]. The Golden Proportion is the most pleasing aesthetic proportion and it has been used in ancient cultures as a proportion basis to compose music, devise sculptures and paintings or construct temples.
The Golden proportion (noted by φ) occurs frequently in geometry, particularly in figures with pentagonal symmetry. The length of a regular pentagon's diagonal is φ times its side. The vertices of a regular icosahedron are those of three mutually orthogonal golden rectangles. The dodecahedron was associated with the fifth element ether or heaven or the cosmos. The 12 faces of this Platonic solid (the dodecahedron) are pentagons containing the Golden proportion. In a pentagram each larger (or smaller) section is related by the Golden Proportion, so that a power series of the Golden Proportion raised to successively higher (or lower) powers is automatically generated [Vajda, 1989].
The Fibonacci Series can be found in the ratio of the number of spiral arms in daisies, in the chronology of rabbit populations, in the sequence of leaf patterns twisted around a branch and many places in nature where self-generating patterns are in effect [www.goldenmuseum.com].
The Fibonacci sequence is a sequence of integer numbers, where every number is the sum of the two previous ones. Thus, beginning with the first two terms 1 and 2 of Fibonacci sequence, we get 1, 1, 2, 3, 5, 8, 13, 21, … .

Fibonacci numbers and the golden mean are used widely in modern theoretical physics. There are many fundamental results in the modern Fibonacci numbers theory, one of them was found in 19^{th} century by the famous French mathematician Binet.

In recent years, a theory of Fibonacci numbers [Dunlap, 1997] was added by many original generalizations of Fibonacci, Lucas numbers and the Golden mean ([El Naschie, 1994], [Spinadel, 1999], [Spinadel, 2002], [Stakhov,2005]).

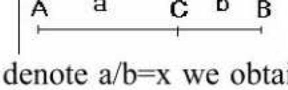
The theory of Fibonacci and Lucas numbers was added by many original generalizations. Some generalizations of Golden Proportion are studied in articles ([Stakhov, 1989], [Stakhov, 1998], [Stakhov, 2005], [Stakhov, 2006], [Stakhov, 2007], [Kocer, 2007]) where A. Stakhov and his collaborators developed a new scientific principle called the Generalized Principle of the Golden Section. In his papers, A. Stakhov studied the properties of hyperbolic Fibonacci and Lucas functions and the continuous functions for the Fibonacci and Lucas p-numbers. His studies was an important step in the development of contemporary Fibonacci numbers theory. Thus, Stakhov and his collaborators defined the m-extension of the Fibonacci and Lucas p-numbers ([Kocer, 2007],[Stakhov,1998]) and they obtained the continuous functions for the m-extension of the Fibonacci and Lucas p-numbers using the generalized Binet formulas.They also introduced a new class of mathematical constants, named the Golden (p,m)-Proportions, which are a wide generalization of the classical golden mean, the golden p-proportions and the golden m-proportions ([Stakhov, 2005], [Stakhov, 2007]). One can find many properties of the Fibonacci and Lucas p-numbers in webpage of the Museum of Harmony and Golden Section ([www.goldenmuseum.com]).

As a generalization of the Golden proportion, the members of the Metallic Mean Family are introduced by the *Vera W. Spinadel* ([Spinadel, 1999]), like the very well known Golden Mean and its relatives, the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others. Also, V. Spinadel generalized the Fibonacci sequence in [Spinadel,1999].

M. S. El Naschie has analyzed the relations existing among the Hausdorff dimension of Cantor sets of higher order and the Golden Mean and the Silver Mean in his paper ([El Naschie, 1994]) and he demonstrated five important theorems, three of them are main theorems (Bijection Theorem, Theorem of the Golden Mean and Generalized Fibonacci Theorem) and two auxiliary theorems (Silver Mean Theorem and Arithmetic Mean Theorem).

Theoretical aspects

In the *Elements*, Euclid of Alexandria (365 BC - 300 BC), defines a proportion derived from a division of a line into what he calls its "extreme and mean ratio". Euclid 's definition states: *"A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser"* ([http://goldennumber.net]):



If we denote a/b=x we obtain the equation

(1) $$x^2 - x - 1 = 0$$

and the positive root of this equation is the golden ratio

(2) $$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots$$

The Golden Rectangle has been known since antiquity as one having a pleasing shape which has the properties the ratio of the length (a+b) to the width (a) always remains in the golden ratio. Thus, if we cut off a square section whose side is equal to the shortest side, the piece that remains is also a golden rectangle:

(3) $a/b = (a+b)/a$

which is equivalent with:

(4) $$x = 1 + \frac{1}{x}$$

If we construct the Golden Rectangle, first we will construct a square ABCD and then, with the center E (the midpoint of AB) and radius EC we draw an arc crossing EB extended at B in F. We construct a perpendicular to AB at F and we extend CD to intersect the perpendicular at G, we can obtain a Golden rectangle AFGD (Figure 1):

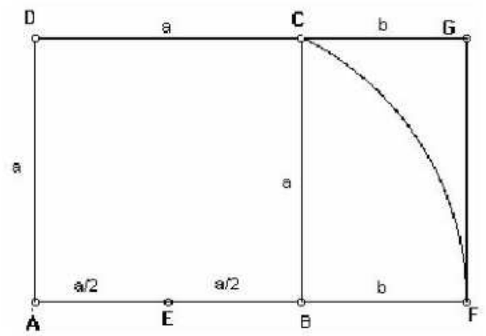


Figure 1: Golden rectangle ([Stakhov,1998])

The Golden rectangle use started as early as with the Egyptians in the design of the pyramids. When the basic golden proportion relationships are used to create a right triangle, it forms the dimensions of the great pyramids of Egypt, with the geometry shown below creating an angle of 51.83 degrees, the cosine of which is the golden proportion. On the other hand, the Greeks used it extensively in the design of the Parthenon and other architecture.

Another remarkable property of the Golden Proportions is its relation with the equiangular spiral. An equiangular spiral can be derived from a golden rectangle: starting with a single Golden rectangle, there is a natural sequence of golden rectangles obtained by removing the leftmost square from the first rectangle, the topmost square from the second rectangle, etc. The length and width of the n-th golden rectangle can be written as linear expressions a+bφ, where the coefficients a and b are always Fibonacci numbers. Many buildings (from Greek temples to European Churches) are proportioned accordingly to the golden mean, a constant which is the ratio of the sides of a rectangle circumscribed about a logarithmic spiral ([Hejazi, 2005]).

The golden function has the form,

(5) $$x = f(t) = \frac{t + \sqrt{t^2 + 4}}{2} \Leftrightarrow x^2 = t \cdot x + 1.$$

We remark that for t=1 then we obtain the equation (1) and the positive solution of this equation is the golden number ϕ .

The problem of the line division in extreme and middle ratio allows the following generalization: let us give the integer non-negative number p (p = 0, 1, 2, 3, …) and divide the line AB by the point C in the following ratio ([Stakhov,2005]):

(6) $$\left(\frac{AB}{BC}\right)^p = \frac{BC}{AC}.$$

If we denote $\frac{AB}{BC} = x$, then according to the equation (1) and from AB = AC + CB, the following algebraic equation follows:

(7) $$x^{p+1} - x^p - 1 = 0$$

for every natural number p.

The equation (7) is called the *golden algebraic equations* and its positive roots are called the *Generalized Golden p-Proportion* ([Stakhov, 2005], [Stakhov, 2007]).

For p=1 in the equation (7) we obtain the equation of the classical Golden Proportion (1).

Modern science widely applies to the well-known Fibonacci {F(n)} and Lucas {L(n)} sequences which result from application of the following recurrence relations [Stakhov, 2005], for n > 1:

(8) $F(n) = F(n-1) + F(n-2)$,

with $F(0) = 0,\ F(1) = 1$

and

(9) $L(n) = L(n-1) + L(n-2)$,

with $L(0) = 2,\ L(1) = 1$.

The Fibonacci numbers appear in arrangements of leaves because the Fibonacci numbers form the best whole number approximations to the Golden proportion. There are some remarkable connections between Golden proportion and Fibonacci numbers: a ratio which stabilizes around the value of the golden number ϕ:

$$(10)\quad \begin{cases} \lim\limits_{n\to\infty}\dfrac{F(n+1)}{F(n)}=\phi, \\ \lim\limits_{n\to\infty}\dfrac{F(n+2)}{F(n)}=\phi^2,... \\ \lim\limits_{n\to\infty}\dfrac{F(n+k)}{F(n)}=\phi^k \end{cases}$$

Another property of connection between the Fibonacci and Lucas numbers [Stakhov, 2007] and the Golden Section is expressed by the well-known mathematical formulas, so-called Binet's formulas :

$$(11)\quad F(n)=\frac{\phi^n-(-1)^n\cdot\phi^{-n}}{\sqrt{5}}$$

and

$$(12)\quad L(n)=\phi^n+(-1)^n\cdot\phi^{-n}$$

In recent years a theory of Fibonacci numbers was added by many original generalizations of Fibonacci, Lucas numbers and the golden mean.

Metallic Means Family (MMF) or Metallic Proportions are another generalization of the golden proportion introduced by V. Spinadel. The members of MMF have, among common characteristics, the property of carrying the name of a metal. Like the very well known Golden Mean and its relatives, the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others ([Spinadel,1999], [Spinadel,2002]). The Fibonacci sequences ([Spinadel,1999]) may be generalized, originating the so called *"generalized secondary Fibonacci sequence"* GSFS, *a, b, pb + qa, p(pb + qa) + qb,* that satisfy relations of the type:

$$(13)\quad G(n+1)=pG(n)+qG(n-1),$$

where p and q are natural numbers.

Let us consider a set of positive irrational numbers, obtained taking $G(0)=G(1)=1$ in equation (13) and considering different natural values for the parameters p and q.

The metallic means family (MMF) is a set of positive eigenvalues of the matriceal equation ([Spinadel, 2002])

$$(14)\quad \begin{pmatrix} G(n+1) \\ G(n) \end{pmatrix}=\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} G(n) \\ G(n-1) \end{pmatrix}$$

for any values of natural numbers p and q.

From (13) we get

$$(15)\quad \frac{G(n+1)}{G(n)}=p+\frac{q}{\dfrac{G(n)}{G(n-1)}}.$$

If the limit:

$$(16)\quad \lim_{n\to\infty}\frac{G(n+1)}{G(n)}=x$$

exists, taking limits in both members we obtain:

$$(17)\quad x=p+\frac{q}{x}\Leftrightarrow x^2-px-q=0,$$

with positive solution is

$$(18)\quad x=\frac{p+\sqrt{p^2+4q}}{2}.$$

Replacing iteratively the value of x of the second term of the relation (14), one remark that:

$$(19)\, x=p+\frac{q}{x}=p+\cfrac{q}{p+\cfrac{q}{p+\cfrac{q}{....}}}\overset{noted}{=}[p,q]$$

Particularly,

- for $p=q=1$ we obtain the Golden Proportion;
- for p = 2 and q = 1 we obtain the Silver Mean;
- for p = 3 and q = 1 we get the Bronze Mean;
- for p= 1 and q = 2, we obtain the Copper Mean;
- for p = 1 and q = 3, we get the Nickel Mean and so on.

Results and discussion

Some generalizations of the Fibonacci and Lucas numbers, called the Fibonacci and Lucas p-numbers are given as follows ([Stakhov, 2005]):

(20) $F_p(n) = F_p(n-1) + F_p(n-p-1)$,

for any natural given p (p = 1,2,3, ...) and n>p+1, with initial conditions

$F_p(0) = 0,\ F_p(1) = F_p(2) = ... = F_p(p+1) = 1$

and for n > p:

(21) $L_p(n) = L_p(n-1) + L_p(n-p-1)$,

with:

$L_p(0) = p+1,\ L_p(1) = L_p(2) = ...L_p(p) = 1$.

In the paper [Kocer, 2007], E. Gokcen Kocer, Naim Tugluand and Alexey Stakhov, introduced a new class of mathematical constants – the Golden (p, m) - proportions, which are a wide generalization of the classical golden mean, the golden *p*-proportions and the golden *m*-proportions.

For given integer $p \geq 0$ and positive real number m > 0 the recurrence relation for the m-extension of the Fibonacci and Lucas p-numbers are given as follows:

(22) $F_{p,m}(n) = m \cdot F_{p,m}(n-1) + F_{p,m}(n-p-1)$,

with initial conditions $F_{p,m}(0) = a_0,\ F_{p,m}(1) = a_1, ... ,F_{p,m}(p) = a_p$,

where $a_1, a_2, ..., a_p$ are integers, real or complex numbers. Positive roots of the characteristic equation for the sequence from (22) are called generalized Golden (p,m)-proportions ([Stakhov,2007]).

We remark that: for m = 1 in (22), the golden (p, 1) - proportions are reduced to the golden p-proportions, positive roots of the characteristic equation and for the case p = 1 the golden (p, m) - proportions are reduced to the Golden m - proportions.

We introduce a generalization of the m-extension for Fibonacci and Lucas p-numbers, given by:

(23) $F_{p,m,n}(k) = m \cdot F_{p,m,n}(k-1) + n \cdot F_{p,m,n}(k-p-1)$,

with initial conditions $F_{p,m,n}(0) = a_0,\ F_{p,m,n}(1) = a_1, ... ,F_{p,m,n}(p) = a_p$,

where $a_1, a_2, ..., a_p$ are real or complex numbers and p, m, n are integer numbers ($m, n, p \neq 0$).

The characteristic equation of the m-extension of the Fibonacci p-numbers has the form:

(24) $x^{p+1} - m \cdot x^p - n = 0$.

Particularly, for the case m = n = p = 1, the Golden (1,1,1)–proportion is reduced to the classical Golden proportion.

We remark that: this equation has (p+1) roots $x_1, x_2, ..., x_p$ (named generalized Golden (p,m,n) - proportions) which verify these conditions:

(25) $$\begin{cases} x_1 + x_2 + ... + x_{p+1} = m \\ \sum_{1 \leq i_1 < ... < i_t \leq p+1} x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_t} = 0, \\ x_1 \cdot x_2 \cdot ... \cdot x_{p+1} = (-1)^p \cdot n \end{cases}$$

$(\forall) t \in \{1, 2, ..., p\}$

Moreover, the roots of the characteristic equation (24) verify that:

(26) $x_1^k + ... + x_{p+1}^k = (x_1 + ... + x_{p+1})^k = m^k$

for every $k \in \{1, 2, ..., p\}$,

and

(25) $x_1^{p+i} + ... + x_{p+1}^{p+i} = m^{p+i} + (p+i) \cdot m^{i-1} \cdot n$

for every $i \in \{1, 2, ..., p+1\}$

If $x_i \neq 0, (\forall) i \in \{1, 2, ..., p+1\}$ we obtain

(26) $$\left(\frac{1}{x_1}\right)^k + \left(\frac{1}{x_2}\right)^k + ... + \left(\frac{1}{x_{p+1}}\right)^k = 0$$

for every $k \in \{1, 2, ..., p\}$,

Geometrically we can obtain generalized Golden (p,m,n) - proportions if we divide a line AB by the point C so that $m \cdot BC < AB$ in the following ratio

(27) $$\left(\frac{AB}{BC}\right)^p = \frac{n \cdot BC}{AB - m \cdot BC}.$$

for every natural numbers p and n. If we denote $\frac{AB}{BC} = x$, then from AB=AC+CB, we obtain the algebraic equation (24).

For $q=s^2$, where s is an integer number, we can construct a "metallic" rectangle: first we construct a rectangle ABCD with the sides $s=\sqrt{q}$ and p. With the midpoint of AB noted by E and the radius

$$EC=\frac{\sqrt{p^2+4s^2}}{2} \quad (28)$$

we can draw an arc crossing EB extended at B in F. Then, we construct a perpendicular to AB at F and we extend CD to intersect the perpendicular at G. Thus, AFGD is a metallic rectangle (Figure 2):

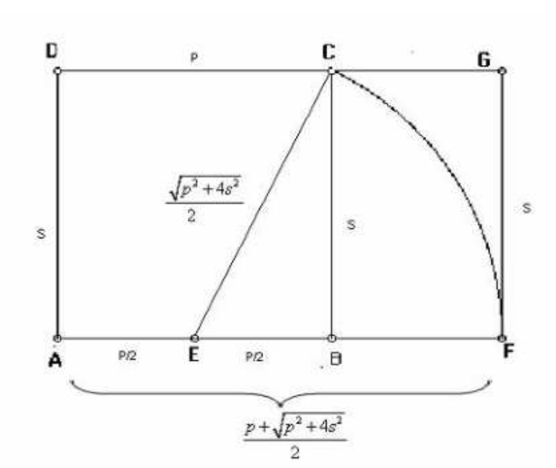


Figure 2: The "metallic" rectangle

As to the Golden rectangle, the proportion of sides (the length and the width) of the rectangle AFGD is the same as to the rectangle BFGC obtained by removing the leftmost rectangle from the first metallic rectangle AFGD.

Many buildings are proportioned accordingly to the golden proportion or a metallic proportions. In the Figure 3 is the Sihăstria Monastery ([www.romanian - monasteries.go.ro]), the holy establishment is situated nearby from the town of Târgu-Neamț - Romania.

The founding of the monastery was Ghedeon (Bishop of Huşi) who completed the construction of church (which was built of wood), a group of monastic cells and a belfry in 1655 ([www.romanian-monasteries.go.ro/neamt/sihastria.htm]).

Using PhiMatrix ([www.phimatrix.com]) - a graphic analysis and design tool that lets us apply the golden proportions to any image - we remark that the golden proportion was used in the design of the Sihăstria Monastery, as a symbol of the beauty and balance in the design of architecture:

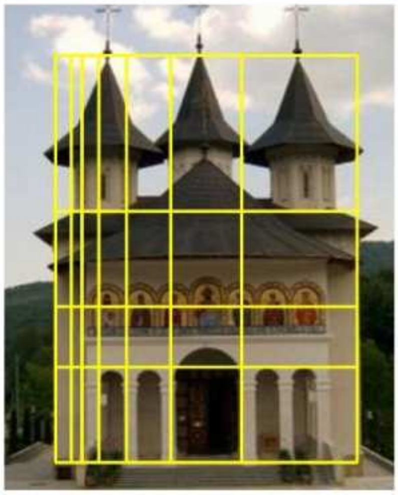


Figure 3: The Golden proportions (drawing by PhiMatrix) in design of Sihăstria Monastery

The golden proportion is not necessarily undertaken consciously, but results from an impression of beauty and harmony in architecture.

Conclusion

The purpose of the present paper is to study generalized Golden Proportions and some applications of the golden proportion in architecture.

We generalize the Golden proportion and we introduce the concept of the golden (p,m,n) – proportions which extend a number of new mathematical constants. We find some relationships between the generalized Golden (p,m,n) - proportions and the class of recurrence numerical sequences named by A. Stakhov the m - extension of Fibonacci p-Numbers and Lucas p-Numbers. The concept of the Golden (p,m,n) – proportions extends a number of new mathematical constants, which can be useful in finding new applications into modern physics. Another generalization of the golden proportion called the metallic means or metallic proportions was developed by the Argentinean mathematician V. Spinadel (starting from 1997 with the first Spinadel's paper in this area) in a series of papers and books ([Spinadel, 1999], [Spinadel, 2002]) where she studied a new generalization of the Golden mean. The Golden Rectangle is a very important shape in mathematics which is often used in art and architecture. The special property of the Golden Rectangle is that the ratio of its length to the width equals to the golden proportion. The reason for the using of golden rectangle in the architecture of beautiful buildings is that it is considered to be one of the most pleasing shapes.

The golden rectangles appear in the architecture of Romanian monasteries, and this fact can be proved by using PhiMatrix (a graphic analysis and design tool) on the image of the Sihăstria Monastery.

As a generalization of the golden rectangle we construct a "metallic" rectangle, which has the analogous property as the golden rectangle: the proportion of the length and the width of the metallic rectangle is the same as to the rectangle obtained by removing the leftmost rectangle with from the first metallic rectangle.

References

DUNLAP, R.A., The Golden ratio and Fibonacci numbers. Singapore: World Scientific; 1997;

HEJAZI, M., Geometry in nature and Persian architecture, Building and Environment 40 (2005) 1413–1427;

KOCER, E. G., TUGLU N., STAKHOV, A., On the m-extension of the Fibonacci and Lucas p-numbers, Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.09.071;

EL NASCHIE, M. S., SILVER, Mean Hausdorff Dimension and Cantor Sets, Chaos,Solitons & Fractals, No. 10, 1994;

EL NASCHIE, M.S., Dimensions and Cantor Spectra, Chaos, Solitons & Fractals, vol. 4, No. 11, 1994;

SPINADEL, V.W., The metallic means family and multifractal spectra. Nonlinear Analysis 1999;36:721–45;

SPINADEL, V.W., The family of metallic means. Vis Math 1999;1(3);

SPINADEL, V.W., The metallic means family and forbidden symmetries. Int Math J 2002;2(3):279–88;

STAKHOV, A.P., Fundamentals of a new kind of mathematics based on the golden section. Chaos, Solitons & Fractals 2005;27(5): 1124–46;

STAKHOV, A.P., The Golden Section in the measurement theory., Comput Math Applic 1989;17(4–6):613–38;

STAKHOV, A.P. , The Golden Section and modern harmony mathematics. Applications of Fibonacci Numbers, vol.7. Dordrecht: Kluwer Academic Publishers; 1998. p. 393–9;

STAKHOV, A.P., The generalized principle of the golden section and its applications in mathematics, science and engineering. Chaos, Solitons & Fractals 2005;26:263–89;

STAKHOV, A.P., Kilic, E.,On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices, Chaos, Solitons and Fractals doi:10.1016/j.chaos.2007.10.007;

STAKHOV, A.P., ROZIN, B., The ''golden'' algebraic equations, Chaos, Solitons & Fractals 2006;27(5):1415–21;

STAKHOV, A.P. , The generalized golden proportions, a new theory of real numbers, and ternary mirror-symmetrical arithmetic, Chaos, Solitons and Fractals 33 (2007), 315–334;

VAJDA, S., Fibonacci & Lucas numbers, and the Golden Section. Theory and applications. Chichester: Ellis Horwood; 1989;

http://goldennumber.net;

www.goldenmuseum.com;

www.phimatrix.com;

www.romanian-monasteries.go.ro.