THE GOLDEN REPRESENTATION OF THE COMPLEX NUMBER AND QUATERNIONS

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Abstract: *The main purpose of this paper is to introduce some new classes of numbers associated with the Fibonacci sequence, Lucas numbers and Quaternions named by us Fibonacci complex sequences, Lucas complex sequences and Fibonacci quaternions respectively and to present a systematic investigation of the various results obtained for these classes of numbers. Also, we establish some golden representations of these classes of numbers using the well known Golden Section.*

Keywords: *Golden Section, Fibonacci complex numbers, Lucas complex numbers, Quaternions.*

Introduction

Golden Section, Fibonacci numbers and quaternions can be seen as a source of fruitful mathematical concepts and theories. The Golden Section and related to it Fibonacci numbers are widely used in geometry of the architectural plans so, can be find in Khufu's Pyramid of Egypt, in the Parthenon in Athens, in Greek sculpture, the "Mona Lisa" by Leonardo da Vinci, paintings by Rafael, etc. (Hoggat,1969), (Stakhov,2001).

The regular pentagon comprises a number of wonderful figures based on the golden section, which are widely used in works of art (for example in ancient Egypt and classic Greece). A graceful figure enclosed into the pentagon is the golden triangle, whose base is the side of the regular pentagon. Another interesting figure related to the Golden section is the golden rectangle (which ratio of the sides equals to the golden ratio). Deriving the square from the golden rectangle, we get the new golden rectangle, which ratio of the sides equals to the golden ratio. If we continue the procedure endlessly we get an infinite sequence of the squares and the golden rectangles.

*The golden proportion defines the dimensions of the human profile (*Meisner, 1997*)*. A human body and all its parts are subordinated to the principle of the golden proportion. It is proved that our hand creates a golden section in relation to your arm, as the ratio of our forearm to our hand is also 1.618, the golden ratio. The human face is based entirely on the golden ratio. In particular, the head forms a golden rectangle with the eyes at its midpoint.

Quaternions were introduced in the mid-nineteenth century by Irish mathematician Sir William Rowan Hamilton (Hamilton,1844) as an extension of complex numbers. Hamilton knew that the complex numbers could be viewed as points in a plane, and he was looking for a way to do the same for points in space (Hamilton,1847). Quaternionic numbers may be represented by points in four-dimensional space. Also, the quaternions can be seen as a tool for manipulating 3-dimensional vectors. Representations of rotations by quaternions are more compact and faster to compute than representations by matrices. For this reason, quaternions are used in computer graphics, control theory (Groß,2001), signal processing, physics, bioinformatics (Karney,2007), and mechanics.

Theoretical aspects

The **Golden Section** arises from the division of the line-segment AB by the point C in the extreme and mean ratio $\frac{AB}{BC}=\frac{BC}{AC}$. It is reduced to the equation $x^2=x+1$, and the positive root of the equation $\phi=\frac{1+\sqrt{5}}{2}$ is named the Golden Section (or the Golden Ratio, or the Golden Mean). The Golden Ratio can be represented in the form of "continued" fraction (Agarwal,1990):

$$\phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\dots}}} \quad (1)$$

or in the form of *"radicals"*:

$$\phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}} \quad (2)$$

The Golden Section is widely used in geometry. Using a regular pentagon, we can find enclosed into the pentagon the golden triangle, whose base is the side of the regular pentagon. The triangle has the vertex angle measuring 36° and the base angles measuring 72° each. Thus, we can find that:

$$\phi=\frac{1+\sqrt{5}}{2}=2\cos 36^o \quad (3)$$

Moreover, the cross points of the diagonals divide them in the golden section and form the new regular pentagon.

The Fibonacci Sequence is the series of numbers: 0, 1, 1, 2, 3, 5, 8, ... In this sequence, the next number is found by adding up the two numbers before it. The Fibonacci numbers $(f_n)_{n=\overline{0,\infty}}$ are, by definition the linear recurrence rule given by:

(4) $f_{n+2}=f_{n+1}+f_n$, with $f_0=0,\ f_1=1$.

This definition of Fibonacci sequence can be extended for every integer number k. Using the recurrence relation (1) it can obtain (Stakhov,2001), for $k\geq 1$:

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}=Q\begin{pmatrix} f_k \\ f_{k-1} \end{pmatrix} \quad (5)$$

where $Q=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is names Q-matrix.

It is obvious that (Stakhov,2006), for $k\geq 1$:

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}=Q^k\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6)$$

and

$$\begin{pmatrix} f_k \\ f_{k-1} \end{pmatrix}=Q^{k-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and these together give:

$$\begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}=Q^k \quad (7)$$

The matrixes Q^k are called the "golden" matrixes (Stakhov,2005).

From the equality (7), using $\det(Q^n)=(\det Q)^n$, we can obtain an important property of Fibonacci numer, named the Cassini formula (Weisstein), (Stakhov,1999), for $k\geq 1$:

$$f_{k+1}f_{k-1}-f_k^2=(-1)^k. \quad (8)$$

The Q-matrices immediately give an important Fibonacci identity:

- from $Q^{n+1}\cdot Q^n=Q^{2n+1}$ we obtain (Stakhov,1999):

$$\begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}\cdot\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}=\begin{pmatrix} f_{2n+2} & f_{2n+1} \\ f_{2n+1} & f_{2n} \end{pmatrix} \quad (9)$$

- from $Q^n\cdot Q^{m-1}=Q^{n+m-1}$ we have:

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}\cdot\begin{pmatrix} f_m & f_{m-1} \\ f_{m-1} & f_{m-2} \end{pmatrix}=\begin{pmatrix} f_{n+m} & f_{n+m-1} \\ f_{n+m-1} & f_{n+m-2} \end{pmatrix} \quad (10)$$

Let L_n the n[th] **Lucas number** defined recursively by (Stakhov,2005):

$$L_{n+2}=L_{n+1}+L_n \text{ with } L_0=2,\ L_1=1. \quad (11)$$

As in the case of Fibonacci numbers, the definition of Lucas sequence can be extended for every integer number k.

First mathematical connection between the golden ratio and Fibonacci and Lucas numbers was established in the 19th century by the well-known French mathematician Binet. Binet's formulas in mathematics are well known as the following group of the formulas (Stakhov,2006):

$$f_n = \frac{\phi^n - (-\frac{1}{\phi})^n}{\sqrt{5}} \quad (12)$$

and

$$L_n = \phi^n + (-\frac{1}{\phi})^n \quad (13)$$

where ϕ is the Golden section.

The terms of the Fibonacci and Lucas sequences have some interesting properties:

- for the odd $n = 2k+1$, we have $f_{2k+1} = f_{-2k-1}$ and for the even $n = 2k$ they are opposite $f_{2k} = -f_{-2k}$
- as for Lucas numbers Ln, it is the contrary, i.e. $L_{2k+1} = -L_{-2k-1}$ and $L_{2k} = L_{-2k}$

It is easy to determine that L_n and F_n are connected each to other by the following Relation (Stakhov,2005):

$$L_n = f_{n+1} + f_{n-1} \quad (14)$$

Using the idea of the Q-matrix, the following generalization of the Q-matrix, called Fibonacci Q_p-matrixes, was introduced in (Stakhov, 1999)

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (15)$$

The other part of the matrix (without the first column and last row) is an identity (p x p) - matrix. For the particularly cases $p \in \{0,1,2\}$, the corresponding Q_p-matrixes have the following form, respectively:

$$Q_0 = (1), \quad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (16)$$

The quaternions can be defined by introducing abstract symbols *i, j, k* which satisfy the rules $i^2 = j^2 = k^2 = ijk = -1$ and the usual algebraic rules *except* the commutative law of multiplication (Zhang, 1997). The elements 1, *i*, *j*, and *k* are a choice of basis for R^4. The set of all quaternions H, which are equal to R^4 is a four-dimensional vector space over the real numbers. Thus, every element of H can be uniquely written as a linear combination of these basis elements, that is, as

$$q = a + bi + cj + dk, \quad (17)$$

where a, b, c, *and* d *are real numbers.*

On H it can define three operations: addition, scalar multiplication, and quaternion multiplication (Farebrother,2003). The sum of two elements of H is defined to be their sum as elements of R^4. Similarly the product of an element of H by a real number is defined to be the same as the product in R^4. Unlike multiplication of real or complex numbers, multiplication of quaternions is not commutative (for example, ij = - ji = − k*). Using the basis 1,* i, j, k *of H makes it possible to write H as a set of quadruples:*

$$H = \{(a,b,c,d) / a,b,c,d \in \mathbb{R}\}$$

The basis elements are: $1 = (1,0,0,0)$, $i = (0,1,0,0)$, $j = (0,0,1,0)$, $k = (0,0,0,1)$.

A number of the form $a + 0i + 0j + 0k$, where a is a real number, is called *real*, and a number of the form $0 + bi + cj + dk$, where b, c, and d are real numbers (possibly all zero), is called *pure imaginary*. If $a + bi + cj + dk$ is any quaternion, then a is called its *scalar part* and $bi + cj + dk$ is called its *vector part*.

If $q = a + bi + cj + dk$ is a quaternion then the *conjugate* of q is the quaternion $\bar{q} = a - bi - cj - dk$.

Conjugation is an involution, meaning that it is its own inverse, so conjugating an element twice returns the original element. Conjugation can be used to extract the scalar and vector parts of a quaternion.
The square root of the product of a quaternion with its conjugate is called its *norm* and is denoted $\|q\|$. It has the formula

$$\|q\| = \sqrt{q \cdot \bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2} . \quad (18)$$

Many operations on vectors can be defined in terms of quaternions, and this makes it possible to apply quaternion techniques wherever spatial vectors arise (for instance, this is true in general relativity, electrodynamics, and 3D computer graphics).

Matrix representation of quaternions (Farebrother, 2003): let I be the 4 x4 identity matrix and let *H, J, K* be 4 x4 matrices with real elements. Then the typical quaternion may be written in matrix form as $Q = aI + bH + cJ + dK$ and the rules of quaternion addition and multiplication will follow from those of matrix addition and multiplication provided that the matrices *H, J, K* satisfy the Hamiltonian conditions: $HH = -I$, $JJ = -I$, $KK = -I$, $HJ = K$, $JK = H$, $KH = J$, $JH = -K$, $KJ = -H$, $HK = -J$.

The quaternion group (Zhang, 1997) can be represented as a subgroup $Q = \{\pm I, \pm H, \pm J, \pm K\}$ of the general linear group $GL_2(\mathbf{C})$, so that: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Results and Discussion

In the same manner like the Fibonacci sequence, we can define a "golden" representation of a complex number given by the sequence

$$z_n = f_{n+1} + i \cdot f_n, \quad n \in \square \quad (20)$$

We can call this sequence as *the Fibonacci complex numbers* $(z_n)_{n=\overline{0,\infty}}$ because it verifies the linear recurrence rule given by:

$$z_{n+2} = z_{n+1} + z_n, \text{ with } z_0 = 1, \ z_1 = 1 + i . \quad (21)$$

This definition of *Fibonacci complex sequence* can be extended for every integer number *n*. Thus, we obtain a sequence of complex numbers: *1, 1+i, 2+i, 3+2i, 5+3i, ...,* $f_{n+1} + i \cdot f_n$,.... (for n>0) and *i, 1-i, -1+2i, 2-3i,...,* $f_{-n+1} + i \cdot f_{-n}$, ... (for n<0). Using a trigonometric representation of z_n we obtain

$$z_n = \sqrt{f_{n+1}^{\ 2} + f_n^{\ 2}} (\cos\theta_n + i \sin\theta_n) \quad (22)$$

where $tg\theta_n = \frac{f_{n+1}}{f_n}$. From (22) we can obtain an interesting property:

$$\lim_{n \to \infty} \theta_n = arctg\phi \quad (23)$$

Using the recurrence relation (21) and the Q-matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ we obtain, for $k \geq 1$:

$$\begin{pmatrix} z_{k+2} \\ z_{k+1} \end{pmatrix} = Q \begin{pmatrix} z_{k+1} \\ z_k \end{pmatrix} = ... = Q^{k+1} \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \quad (24)$$

with "golden" matrixes $\begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix} = Q^{k+1}$.

Using the Binet's formula (12) for Fibonacci and the equality (20) we can obtain:

$$z_n = \frac{\phi^{n+1} - (-\frac{1}{\phi})^{n+1}}{\sqrt{5}} + i \frac{\phi^n - (-\frac{1}{\phi})^n}{\sqrt{5}} \quad (25)$$

In the same manner as in the case of the Fibonacci complex numbers, we can the *Lucas complex number* defined recursively by:

$$z_n = L_{n+1} + i \cdot L_n \quad (26)$$

with $z_0 = 1 + 2i$, $z_1 = 3 + i$

As in the case of Fibonacci numbers, $z_{n+2} = z_{n+1} + z_n$ and the definition of

Lucas sequence can be extended for every integer number n.

Using the Binet's formula (13) for *Lucas numbers* and the equality (26) we can obtain:

$$z_n = \phi^{n+1} + (-\frac{1}{\phi})^{n+1} + i\left(\phi^n + (-\frac{1}{\phi})^n\right) \quad (27)$$

The terms of the *Fibonacci and Lucas complex sequences* have some interesting properties:

- for the odd $n = 2k+1$, we have $z_{2k+1} = \overline{z}_{-2k-1}$ and for the even $n = 2k$ we obtain $z_{2k} = -\overline{z}_{-2k}$
- as for Lucas numbers L_n, it is the contrary, i.e. $z_{2k+1} = -\overline{z}_{-2k-1}$ and $z_{2k} = \overline{z}_{-2k}$

If $q = a + bi + cj + dk$ is a quaternion and we replaces the coeficients a,b,c,d with the consecutive elements of the Fibonacci sequence, we can define a "golden" representation of a quaternion, given by the sequence

$$(28)\, q_n = f_{n+3} + i \cdot f_{n+2} + j \cdot f_{n+1} + k \cdot f_n, \quad n \in \square$$

We can call this sequence as *the Fibonacci quaternions* $(q_n)_{n=\overline{0,\infty}}$ because it verifies the linear recurrence rule given by:

$$q_{n+2} = q_{n+1} + q_n, \quad (29)$$

with

$$q_0 = 2 + i + j, \quad q_1 = 3 + 2i + j + k$$

The matrix representation of quaternions is given by

$$Q_n = f_{n+3}I + f_{n+2}H + f_{n+1}J + f_nK, \quad n \in \square \quad (30)$$

and using the matrixes from (19) we obtain:

$$Q_n = f_{n+3} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f_{n+2} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + f_{n+1} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + f_n \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (31)$$

and from (31) we find:

$$Q_n = \begin{pmatrix} z_{n+2} & z_n \\ -\overline{z}_n & \overline{z}_{n+2} \end{pmatrix} \quad (32)$$

Conclusion

The main result of the present paper is to establish a golden representations for some new classes of numbers associated with the Fibonacci sequence, Lucas numbers and quaternions. Using these classes of numbers we find a deep connection between the Golden Section and Fibonacci, respectively Lucas numbers and we present a systematic investigation of the various results which are similar with the properties of Fibonacci and Lucas numbers.

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