SOME CONSIDERATIONS ON FRACTAL DIMENSION

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Abstract

Traditional mathematical tools used for analysis of fractals allow us to distinguish results of self-similarity processes after a finite number of iterations. Fractal dimension has often been applied as a parameter of complexity, related to surface roughness, or for classifying textures or line patterns. Fractal dimension can be estimated statistically, if the pattern is known to be self-similar. Our goal is to provide basic concepts of topological and fractal dimension, to give some examples of common fractal objects and compute their fractal dimension.
Key words: *fractal geometry, fractal dimension, self-similarity.*

Introduction

During last decades fractals have been intensively studied and applied in various fields. Their mathematical analysis very often continues to have mainly a qualitative character and there are not so many tools for a quantitative analysis of their behavior after executing infinitely many steps of a self-similarity process of construction.

Fractals are the place where Mathematics, Science and Art come together. Fractal geometry is becoming increasingly more important in the study of image characteristics. There are many mathematical structures that are fractals; e.g. Sierpinski triangle, Koch snowflake, Mandelbrot set, etc. Fractals also describe many real-world objects, such as clouds, mountains and coast lines that do not correspond to simple geometric shapes. In 1975, B. B. Mandelbrot introduced the term *fractal (*from the Latin *fractus,* meaning broken) to characterize spatial or temporal phenomena that are continuous but not differentiable*)*. Fractal objects and processes are said to display "self-similar" (or self-affine) properties (Hastings, 1993) i.e., small subsets of the object resemble (statistically) the whole. Fractal properties include scale independence, self-similarity, complexity, and infinite length or detail. A mathematical fractal is defined as any series for which the

Hausdorff dimension (a continuous function) exceeds the discrete topological dimension (Tsonis, 1987).

From the mathematical point of view, Becker (1983) classified fractals into three major categories. The first, **iterated function system** (like Koch Snowflake) can generate a fractal from any set of vectors or any defined curve. The second is **the complex number fractals,** which can be two-dimensional, three dimensional or multiple-dimensional (the Mandelbrot set and Julia set). The third is **orbit fractals,** which are generated by plotting an orbit path in two or three-dimensional space (the Bifurcation orbit, Lorenz Attractors).

In order to provide a basis for the applications of fractals to experimental results, this paper focuses on the fractal dimension.

Experimental

Methods to measure fractal dimension

A fractal object has two basic characteristics: infinite detail at every point and a certain self-similarity between the object parts and the overall features of the object.

The self-similarity properties of an object can take different forms, depending on the representation we choose for the fractal. In mathematics, a class of complex geometric shapes that commonly exhibit the property of self-similarity, such that a small portion of it can be viewed as a reduced scale replica of the whole.

Because of the infinite detail inherent in the construction procedures, a fractal object has no definite size. When the detail in an object description is included more, the dimensions increase without limit, but the coordinate extents for the object remain bound within a finite region of space.

We can use the notion "dimension" in two senses: the three dimensions of Euclidean space ($D \in \{1, 2, 3\}$) and the number of variables in a dynamic system. Fractals, which are irregular geometric objects, require a third meaning: the Hausdorff Dimension.

In physical systems, the fractal dimension reflects some properties of the system. The physical characteristics of some bodies are related to the fractal dimension of their surfaces.

For example, the growth pattern of bacteria has a fractal dimension of 1.7, and the fractal dimension of clouds is 1.30 to 1.33; for snowflakes it is 1.7, for coastlines in South Africa or Britain, 1.05 to 1.25, and for woody plants and trees, 1.28 to 1.90 (Taylor, 2002).

Biologists have traditionally modeled nature using Euclidean representations of natural objects or series. Examples include the representation of conifer trees as cones, cell membranes as curves or simple surfaces, etc. Biological systems and processes are typically characterized by many levels of substructure, with the same general pattern repeated in an ever-decreasing cascade.

In medicine, fractal dimensions have been found for various biomolecules such as DNA and proteins. Self-similarity has been found also in DNA sequences. In the opinion of some biologists fractal properties of DNA can be used to resolve evolutionary relationships in animals.

Scientists discovered that the basic architecture of a chromosome is tree-like; every chromosome consists of many "mini-chromosomes" and therefore, can be treated as fractal. For a human chromosome, for example, a fractal dimension *D* equals 2.34 (between the plane and the space dimension).

The fractal dimension of lysozyme (egg-white) is 1.614; for hemoglobin it is 1.583, and for myoglobin 1.728 (Iannaccone, 1996). The fractal dimension of the perimeter of surface cell sections has been used to distinguish healthy cells from cancerous cells (Bauer, 1999).

In analytical chemistry, the fractal dimension is used as a tool to characterize chemical patterns and problems of sample homogeneity (Danzer, 2000).

Topological dimension is always a nonnegative integer. To obtain a finer measurement of the complexity of a space, it is essential to have a way to "measure" things. Therefore, we will assume that our space X is a metric space with a distance function d. Thus, we cross the boundary from topology to geometry.

We start with simple geometric figures that we know how to measure. The length of the line segment form (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ according to Pythagoras' formula. Then, the approximate length of the curve is the sum of the lengths of the line segments. If the same limiting value L of the sum of the lengths of the line segments always occurs, we say the curve is *rectifiable* and that the length of the curve is L. For areas, we use the same sort of reasoning approximating the region by a union of small rectangles.

This generalized treatment of dimension is named after the German mathematician, Felix Hausdorff. Historically, Hausdorff (1919) introduced the term of noninteger dimension and with the given definition the age of fractality has begun.

The amount of variation in the structure of a fractal object can be described with a number D, called the fractal dimension, which is a measure of the roughness or fragmentation of the object. There are some methods to generate a fractal objects. As one method of them, there is an iterative procedure that uses a selected value for D.

A set that can be assigned a fractal dimension is called a fractal set. One can determine the fractal dimension of the set by observing optimal covering systems of fractal sets with decreasing diameters. It should be mentioned that several different definitions of fractal dimension were created since Hausdorff's paper.

For self similar sets most of these definitions lead to the same dimension number (Sandau, 1996).

If a set is given in a binary image, one can always measure its fractal dimension. Several methods use a regression along the range of possible magnifications in the image (Falconer, 1994). An often-used method of this type is the so called box-counting method (denoted by *bcm*). In bcm, the number of boxes of a regular grid with boxes of side length L, intersecting the set of interest, are counted.

The logarithm of this number is plotted versus *log(L)* in a so-called "log-log-plot". In case of self-similar sets the graph has globally a constant slope which is directly related to the fractal dimension. To calculate the box-counting dimension, we need to place a picture on a grid. The *x-axis* of the grid is s where s=1/(width of the grid). For example, if the grid is 240 blocks tall by 120 blocks wide, s=1/120. Then, count the number of blocks that the picture touches. Label this number N(s).

Now, resize the grid and repeat the process. Plot the values found on a graph where the x-axis is the *log(s)* and the y-axis is the *log(N(s))*. Draw in the line of best fit and find the slope. The box-counting dimension measure is equal to the slope of that line. The Box-counting dimension is much more widely used than the self-similarity dimension since the box-counting dimension can measure pictures that are not self-similar (and most real-life applications are not self-similar).

One can be assumed that for any fractal object (of size P, made up of smaller units of size p), the number of units (N) that fits into the larger object is equal to the size ratio (P/p) raised to the power of D. Thus, if we take an object of dimension D and reduce its linear size by P/p in each spatial direction, its measure (length, area, or volume) would increase to $N=(P/p)^D$ times the original, thus log(N) = D log(P/p). If we solve for D we obtain

$$D = \frac{\log(N)}{\log(P/p)}$$

We remark that D is not necessarily an integer, as it is in Euclidean geometry.

Results and Discussion

Examples of geometric objects with non-integer dimensions

1. Cantor's set dimension

We consider a line segment of unit length and we remove its middle third. Then, we remove the middle thirds from the remaining two segments. We could continue this construction through infinitely many steps. What remains after infinitely many steps is a remarkable subset of the real numbers called, the Cantor set.

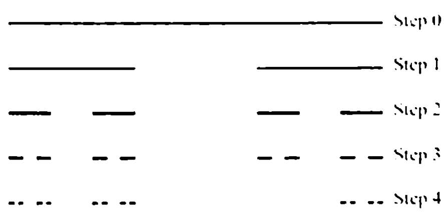

Fig.1: Cantor's set

After that, all the lengths of the intervals we removed add up to 1, exactly the length of the segment we started with:

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots+\frac{2^n}{3^{n+1}}+\ldots=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n=1$$

Fractal dimension of Cantor Set: at each step, one line segment is divided into 2 self-similar pieces with scaling factor 1/3, hence fractal dimension of Cantor set is $D=-\frac{\log 2}{\log(1/3)}\cong 0,63$

2. *Fractal dimension of Koch snowflake curve*

We begin with a straight line and we divide it into three equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length as the segment being removed. The first iteration for the Koch curve consists of taking four copies of the original line segment, each scaled by **r** = 1/3. Two segments must be rotated by 60°, one counterclockwise and one clockwise. Now we repeat, taking each of the four resulting segments, dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle (the red segments in the bottom figure).

The Koch curve is the limiting curve obtained by applying this construction an infinite number of times. The "limit curve" defined by repeating this process an infinite number of times is called the ***Koch's snowflake curve***, named after Niels Fabian Helge von Koch (Sweden, 1870-1924). Koch constructed his curve in 1904 as an example of a non-differentiable curve

The length of the intermediate curve at the n^{th} iteration of the construction is $(4/3)^n$, where $n = 0$ denotes the original straight line segment. Therefore, the length of the Koch curve is infinite and the length of the curve between any two points on the curve is also infinite since there is a copy of the Koch curve between any two points.

For a proof that this construction does produce a "limit" that is an actual curve, i.e. the continuous image of the unit interval (Edgar, 1990).

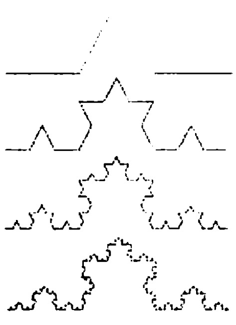

Fig. 2: Koch Snowflake curve

Using only line segments that are 3 centimeters long (P), we make a simple Koch's Curve: 1 segment of 3 centimeters per segment. If you take that to the next level and use line segments which are 1 centimeter long (p), you use 4 line segments of 1 cm, then 12 segments, etc.

By cutting the length of the line segments by one third (P = 3, p = 1, P/p = 3), the number of line segments used (N) goes up four times. That means N = 4, P/p = 3, so fractal dimension of Koch's snowflake curve is D = ln 4 / ln 3≅ 1.26….

3. Fractal dimension of Sierpinsky triangle

Like other fractal objects, Sierpinsky triangle can be obtained through an algorithm.

The algorithm is as follows: Start with any triangle in a plane. An equilateral triangle with a base parallel to the horizontal axis is used most commonly for this purpose.

Scale the triangle by half, take three such triangles, and position them so that each triangle touches the two other triangles at a corner. Repeat this step with each of the smaller triangles.

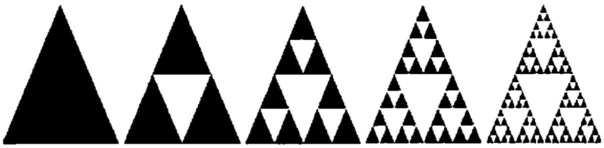

Fig. 3: Sierpinsky Triangle

On every iteration, we replace each triangle with three similar triangles using scaling factor = 1/2. Hence the fractal dimension D can be calculated as $D = \dfrac{\ln 3}{\ln 2} \cong 1{,}58...$

Conclusion

Fractal geometry permits generalization of the fundamental concepts of dimension and length measurement.

It is important to recognize that while Euclidean geometry is not realized in nature, neither is strict mathematical fractal geometry.

Fractal Dimension allows us to measure the degree of complexity by evaluating how fast our measurements increase or decrease as our scale becomes larger or smaller.

In applications to natural science, one usually takes the point of view that the fractals that occur in nature are well-behaved with respect to the calculation of their dimension.

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