SOME RELATIONSHIPS BETWEEN INVARIANTS OF THE SUBMANIFOLDS IN THE RIEMANNIAN MANIFOLDS

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Rezumat:

Una dintre problemele fundamentale ale teoriei subvarietăților este stabilirea unor relații între invarianții intrinseci și extrinseci ai subvarietăților. Scopul principal al acestui articol este de a prezenta câteva aplicații referitoare la egalități și inegalități, folosind acești invarianți ai unor subvarietăți arbitrare în varietăți Riemanniene.

Abstract:

One of the most fundamental problems in submanifold theory is to establish simple relationships between intrinsic and extrinsic invariants of the submanifolds. The main purpose of this article is to present several applications of equalities and inequalities for arbitrary submanifolds in Riemannian manifolds involving these invariants.

Résumé:

L'un des problèmes fondamentaux de la théorie des subvariétes est d'établir des relations entre des invariants intrinsèque et extrinsèque sur les variétés. Le but principal de cet article est de présenter plusieurs applications et des égalités et inégalités pour les subvariétes arbitraires in variétés Riemannien, réalisés avec ces invariants.

Abstrakt :

Ein bedeutenstes Thema in der Theorie der Untervarietäten ist die Festsetzung einiger einfacher Relationen zwischen der eigentlichen der Untervarietäten. Das Hruftthema dieses Artikels ist das Vorbringen einiger Verwendungen und Egalitaten, und Inegalitaten der Riemannienen Untervarietäten, die mit Hilfe dieser Urbilder der Untervarietäten realisiert worden sind.

Introduction

The theory of submanifolds was started with curvature of plane curves. For a surface in Euclidian 3-space one has the two important quantities (the mean curvature and the Gauss curvature).The mean curvature is an extrinsec invariant which measures the surface tension of the surface arisen from the ambient space. The Riemannian geometry forms the theory of modern differential geometry. Riemannian invariants are the intrinsic characteristic of the Riemannian manifolds and they affect the behavior in general of the Riemannian manifolds. Curvature invariants are the most natural and they play key roles in physics. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of space time. All sorts of shapes from soap bubbles to red cells, seems to be determined by various curvatures ([1],[2]). The differential geometry of surfaces in Euclidian 3-space was generalized to the differential geometry of higher dimensional submanifolds of Riemannian manifolds.

Preliminaries

For the submanifold M in N we denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and N respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [1])

$$(1.1)\quad \overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

$$(1.2)\quad \overline{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for vector fields X, Y tangent to M and ξ normal to M, where h denotes the second fundamental form, D the normal connection, and A is the shape operator of the submanifold. It is well known that

$$(1.3)\quad g(A_\xi X,Y)=\bar{g}(h(X,Y),\xi)$$

where $\bar{g}$ is the metric on N and g is the induced metric on M.

The tangent bundle of N, restricted to M is the direct sum of the tangent bundle T(M) of M and the normal bundle $T^{\perp}(M)$:

$$(1.4)\quad T(N)_{/M}=T(M)\oplus T^{\perp}(M).$$

Definition 1: A submanifold M is said to be *totally geodesic* if the second fundamental form h vanishes identically (that is h=0).

Definition 2: For a normal section ξ on M, if A_ξ is everywhere proportional to the identity transformation I (that is A = λI for some function λ on M), then ξ is called an *umbilical section* and M is said to be umbilical with respect to ξ. If the submanifold M is umbilical with respect to every local normal section in M, then M is said to be *totally umbilical.*

Let M be an n-dimensional submanifold of a Riemannian m-manifold N. We choose a local field of orthonormal frame $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ in N such that, restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M and $e_{n+1}, \dots, e_m$ are normal to M. Let $A_\alpha = A_{\xi_\alpha}$ ($\alpha \in \{n+1,\dots,m\}$).

Definition 3: The *mean curvature vector* H at P is a normal vector at P which is independent of ortonormal basis,

$$(1.5)\quad H=\frac{1}{n}\sum_{\alpha}^{m-n}(traceA_\alpha)\xi_\alpha=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i)$$

Definition 4: A submanifold M is called *a pseudoumbilical submanifold* if there exists a function λ on the submanifold M such that:

$$(1.6)\quad \bar{g}(h(X,Y),H)=\lambda g(X,Y)$$

for any vector fields X and Y on M.

Some relationships between invariants

Remark 1 ([1]): Every minimal submanifold is a pseudoumbilical submanifold, and for a pseudoumbilical submanifold we have $\lambda=\bar{g}(H,H)$.

Remark 2 ([1]): A totally umbilical submanifold is totally geodesic (i.e h=0) if and only if it is minimal (i.e. H=0).

Remark 3 ([1]): For a unit normal vector ξ of M at point P, the second fundamental tensor with respect to ξ, A_ξ is self adjoint hence, there exist ortonormal vector $e_1, \dots, e_n$, of M in P which are

the eigenvectors of A_ξ (that is $A_\xi(e_i) = h_i e_i$ for real numbers h_i which are the principal curvatures and $e_1, \ldots, e_n$ which are principal direction of the normal direction ξ).
The curvature tensor of the Riemannian manifold N is:

$$(1.7)\quad \overline{K}(\overline{X},\overline{Y})\overline{Z} = \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z} - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}$$

where $\overline{X},\overline{Y},\overline{Z}$ are the vector fields on the submanifold M. Let X, Y, Z, W be vector fields on the submanifold M, then we have *the equation of Gauss*:

$$(1.8)\quad \overline{K}(X,Y;Z,W) = K(X,Y;Z,W) + \overline{g}(h(X,Z),h(Y,W)) - \overline{g}(h(X,W),h(Y,Z))$$

where $K(X,Y;Z,W) = g(K(X,Y)Z,W)$ and $\overline{K}(X,Y;Z,W) = \overline{g}(\overline{K}(X,Y)Z,W)$.

If M is a **hypersurface** of N (i.e. M is of codimension 1 in N), then $D_X\xi = 0$. The curvature tensor K of the hipersurface M is describe in term of A (shape operator) and the curvature tensor $\overline{K}$ of the ambient space by the Gauss equation, which can be written as:

$$(1.9)\quad K(X,Y)Z = (\overline{K}(X,Y)Z)^T + g(AX,Z)AY - g(AY,Z)AX$$

for all tangent vector field X,Y,Z tangent on M, and "T" denotes projection on $\chi(M)$. The Codazzi equation of the hypersurface describes the normal component of $\overline{K}(X,Y)Z$ in terms of the derivative of the shape operator. It is given by:

$$(1.10)\quad \overline{g}(\overline{K}(X,Y)Z,\xi) = g((\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y, Z)$$

where $\nabla_X A_\xi$ denotes the covariant derivative of A.
If the ambient space has constant sectional curvature, then $\overline{K}(X,Y)Z$ is tangent to M for every X,Y,Z tangent vector fields on M. Thus, (1.10) becomes

$$(1.11)\quad (\nabla_Y A_\xi)X = (\nabla_X A_\xi)Y$$

As is well known, the shape operator A is a self-adjoint linear operator in each tangent plane $T_P M$ and its eigenvalues $h_1(P), h_2(P) \ldots, h_n(P)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebric invariants, named the
r-th mean curvature in P, given by:

$$(1.12)\quad H_r(P) = \frac{(-1)^r}{C_n^r} \sum_{i_1<\ldots<i_r} h_{i_1}(P) h_{i_2}(P) \cdot \ldots \cdot h_{i_n}(P), \quad 1 \le r \le n.$$

In particular, $H_1 = \frac{1}{n} trace(A) = | H |$ is the mean curvature of M, which is the main extrinsic curvature of the hypersurface.

On the other hand, when r = 2, H_2 defines a geometric quantity which is related to the intrinsic scalar curvature of the hypersurface. Indeed, from [5], it follows from the Gauss equation that the Ricci curvature of M is given by:

$$(1.13)\quad Ric(X,Y) = \overline{Ric}(X,Y) - \overline{g}(\overline{K}(X,\xi)Y,\xi) + n\,|\,H\,|\,g(A_\xi X,Y) - g(A_\xi X, A_\xi y)$$

for $X,Y \in \chi(M)$, where $\overline{Ric}$ stands for the Ricci curvature of the ambient space N. Therefore, the scalar curvature ρ of the hypersurface M is:

$$(1.14)\quad \rho = trace(Ric) = \overline{\rho} - 2\overline{Ric}(\xi,\xi) + n(n-1)H_2$$

If the ambient space has constant sectional curvature c, then we have:

$$(1.15)\quad \rho = n(n-1)(c + H_2).$$

Let $S_r(P) = (-1)^r C_n^r \cdot H_r(P)$ denote the r-th elementary symmetric function on the eigenvalues of A in P. Then, we have:

$$(1.16)\quad \det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r}$$

where $S_o = 1$ by definition.

If $P \in M$ and $\{e_i\}_{i=\overline{1,n}}$ is a basis of $T_P M$, formed by eigenvectors of A, with corresponding eigenvalues $\{\lambda_i\}_{i=\overline{1,n}}$, where $\lambda_i = h_i(P)$, then it is immediate to check that

$$(1.17)\quad S_1^{\ 2} - 2S_2 + trace(A^2) = 0.$$

If the ambient space N has constant sectional curvature "c" then, from the Gauss equation we have

$$(1.18)\quad n(n-1)(c - \rho) = 2S_2$$

where ρ is the scalar curvature of M.

Such function satisfy a very useful set of algebraic inequalities, usually reffered to as Newton's inequalities.

Theorema ([6]): Let n > 1 an integer and $\lambda_1, \lambda_2, ..., \lambda_n$ be real numbers. Then, we have:

(1) For $1 \le r < n$, one has $H_r^2 \ge H_{r-1} H_{r+1}$. Moreover, if equalities happens for r=1 or for some $1 < r < n$ with $H_{r+1} \ne 0$, in this case, then $\lambda_1 = \lambda_2 = ... = \lambda_n$.

(2) If $H_1 = H_2 = ... = H_r > 0$, for some $1 < r \le n$, then $H_1 \ge \sqrt{H_2} \ge \sqrt[3]{H_3} \ge ... \ge \sqrt[r]{H_r}$. Moreover, if equality happens for some $1 \le j < r$, then $\lambda_1 = \lambda_2 = ... = \lambda_n$.

(3) If, for some $1 \le r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \le j \le n$. In particular, at most (r-1) of λ_i are different from zero.

For the proof, recall that if a polynomial $f \in \Re[X]$ has k>0 real roots, then its derivative f' has at least k-1 real roots. In particular, if all roots of f are real, then the same is true for all roots of f'.

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