ON THE GEOMETRY OF FRACTAL SHAPES

Cristina-Elena HREŢCANU,
"Ştefan cel Mare" University-Suceava, Food Engineering Faculty,
cristinah@usv.com

Rezumat
Fractalii sunt figuri geometrice generate de o operaţie care este efectuată în mod repetat, producţia unei repetiţii fiind modul de pornire al repetiţiei următoare. În multe cazuri, un fractal poate fi descris printr-un proces recursiv (repetitiv). Una dintre proprietăţile interesante ale fractalilor este auto-similaritatea, care înseamnă că fiecare parte mică a figurii fractale poate fi reprodusă exact în proporţiile mari ale acesteia. Principiul părţii asemănătoare cu întregul este cuprins şi realizat aproximativ în natură. Astfel, suprafaţa unui munte, forma copacilor sau a ferigilor etc, sunt fractali care pot fi generaţi de computer folosind un algoritm recursiv.
Cuvinte cheie: geometrie fractală, dimensiune fractală, recursivitate.

Résumé
Les fractals sont des figures géométriques produites par une opération effectué d'une manière répétitive, la production d'une répétition étant le mode de départ de la suivante. Dans plusieurs cas, un fractal peut être décrit par un procès récursif (répétitif). Une des propriétés intéressantes des fractals est l'auto-similarité, qui signifie que chaque petite partie de la figure fractale peut être reproduite exactement dans les grandes proportions de celle-ci. Le principe de la partie similaire à l'entier est contenu et réalisé approximativement dans la nature. Ainsi, la surface d'une montagne, la forme des arbres ou des fougères, etc., sont des fractals qui peuvent être faits à l'aide de l'ordinateur en utilisant un algorithme récursif.
Mots clef: géométrie fractale, dimension fractale, l'auto-similarité.

Abstract
Fractals are geometrical figures that are generated by starting with a very simple pattern that grows through the application of rules. In many cases, the rules to make the geometric figure grow from one stage to the next involve taking the original figure and modifying it or adding to it. This process can be repeated recursively an infinite number of times. One interesting property that fractals can have is that of self-similarity. What self-similarity means is that each small portion, when magnified, can reproduce exactly a larger portion. The surface of a mountain, trees and ferns, etc. are fractal in nature and can be modeled on a computer using a recursive algorithm.
Key words: fractal geometry, fractal dimension, self-similarity.

Introduction

Fractal images are shapes showing self-similarities at smaller and smaller scales. Many shapes in nature display this same quality of self-similarity ([5]). Clouds, ferns, coastlines, mountains, etc. all possess this feature. Fractional dimension means that a shape is neither 1, 2 or 3 dimensional, but actually may fall between these numbers, being composed of fractions.

Bertrand Russell wrote in 1907 that *"Mathematics, rightly viewed, possesses not only truth, but supreme beauty"*, which applies so well to fractals. They were discovered by the mathematician Benoit B. Mandelbrot in the 1970's and Mandelbrot's fractal geometry provides a mathematical model for many complex forms found in nature such as shapes of coast lines, mountains, galaxy clusters, and clouds. Euclidean geometry allowed us to study and understand regular shapes: polyhedrons and circles, but very little in nature is so regular. The term "**fractal**" derived from the Latin verb ***frangere***, meaning to break or fragment. In the language of calculus, many curves are differentiable (for example, the trajectory of the shell is a classic example) but fractals, like bumps of broccoli, are not differentiable: the closer you come, the more detail you see. Infinity is implicit and invisible in the computations of calculus but explicit and graphically manifest in fractals.

Fractals have come up as an important question when British map makers discovered the problem with measuring the length of Britain's coast. Looking at really detailed maps, the coastline was over double the original. This is a property of fractals (a finite area being bounded by an infinite line).

Fractals have always been associated with the term chaos. For many chaologists, the study of chaos and fractals is more than just a new field in science that unifies mathematics, theoretical physics, art, and computer science. It is the discovery of a new geometry, one that describes the boundless universe we live in. Today, many scientists are trying to find applications for fractal geometry, from predicting stock market prices to making new discoveries in theoretical physics. Fractals are the place where math, science and art come together. Fractal geometry is becoming increasingly more important in the study of image characteristics. Some of the major differences between fractal and Euclidean geometry are: the recognition of fractal is very modern, they have only been studied in the last 50-60 years compared to Euclidean geometry which goes back over 2000 years and secondary. On the other hand, Euclidean geometry provides a good description of man made objects whereas fractals are required for a representation of naturally occurring geometries. There are many mathematical structures that are fractals; e.g. Koch snowflake, Mandelbrot set, etc

Methods to created fractals

Examining fractals, we were able to classify them into two major categories ([4]):

- the first is line or vector fractals, from the drawing method point of view (these are generated from the replacement of a group of vectors);
- the second are fractals that are generated as a group of points in the complex plane

From the mathematical point of view, we can classify fractals ([2]) into three major categories:

1. The first, **iterated function system** (like Koch Snowflake). Iteration means to repeat a process over and over again.
2. The second is **the complex number fractals**. They can be two-dimensional of three dimensional or multiple-dimensional (the Mandelbrot set and Julia set)
3. The third is **orbit fractals**. They are generated by plotting an orbit path in two or three-dimensional space (the Bifurcation orbit, Lorenz Attractors)

A common property of fractal object is self-similar. Thus, a fractal is a shape made of parts similar to the whole in some way. One deterministic approach to generate or to approximate a mathematical fractal object is to use **Iteration function system (IFS)** ([5]). An IFS is a family of specified contraction mappings that map a whole object into the parts, unionize all the parts and iteration of these mappings will result in convergence to an invariant set ([6]). From the viewpoint of geometric modeling, unionization in IFS is regularized in order to maintain the validity of the operands in further teration. Thus, the equation of Regularized Iterated Function System generation is

$$f^k(S) = f(f^{k-1}(S)),\ k = 2,3,\ldots$$

where S is the self-similar segment, f is the iterated function, and k is the iteration number.

Benoit Mandelbrot, an employee of IBM, thought about writing a program with a formula such as, $z_n^2 + c$ and then running it on one of IBM's computers. The Mandelbrot set is created by a general technique where a function of the form $z_{n+1} = f(z_n)$ is used to create a series of a complex variable ([6]). In the case of the Mandelbrot the function is

$$f(z_n) = z_n^2 + c,$$

for any $n \in ¥$ and c is a constant. This series in generated for every initial point z_o =c on some partition of the complex plane. This is known as the recursion law. This specific equation will form a fractal known as the Julian set. In this equation, c is a complex number. Fractal equations are iterative (thus, the result of one calculation of the fractal equation becomes the z input to the next calculation). Over repeated evaluations of a fractal equation, values for each point in the (x, y) coordinate space either converge at single points, move toward the (0, 0) origin point, or move toward infinity. If the point never leaves the screen (of the computer), then we go back to the first coordinates for that point, and make the pixel there a certain color. The points that never leave the screen are all colored the same color. If eventually the point does leave the screen, however, than we count how many times we had to iterate our function to make it leave, and use that number to color the pixel at the original coordinates. Thus, the diverse colors in fractal plots reflect the rate of this movement for each point. For example, if the point leaves after one iteration, it is assigned a color. Every point after, that leaves the graph after one iteration, is that same color. All points that leave after two iterations will be assigned a different color. Every point that never leaves the screen is assigned one color, usually black.

To iterate $z^2 + c$, we begin with a *seed* for the iteration. This is a (real or complex) number which we denote by z_0. Applying the function $z^2 + c$ to z_0 yields the new number $z_1 = (z_0)^2 + c$ .Now, we iterate, using the result of the previous computation as the input for the next.

That is $z_2 = (z_1)^2 + c$, $z_3 = (z_2)^2 + c, \ldots$ $z_n = z_{n-1}^2 + c, \ldots$The list of numbers $z_0, z_1, z_2, \ldots z_n \ldots$ generated by this iteration is called the *orbit* of z_0 under iteration of $z^2 + c$.

For example, if the constant c = 1 and we choose the seed $z_0 = 0$, the orbit is $z_0 = 0$, $z_1 = 1$, $z_2 = 2$, $z_3=5, \ldots$, $z_n = z_{n-1}^2 + 1, \ldots$ and we see that this orbit tends to infinity. As another example, for c = 0, the orbit of the seed $z_0 = 0$, we remark that $z_n = 0$, for any $n \in ¥$ and we see that this orbit is constant 0. If we now choose c = -1 and for the seed $z_0=0$, the orbit is $z_0 = 0$, $z_1 = -1$, $z_2 = 0$, $z_3 = -1, \ldots$ We see that the orbit bounces back and forth between 0 and -1, a *cycle of period 2*.

Under iteration of $z^2 + c$, when the orbit does not go to infinity, it may behave in a variety of ways. The Mandelbrot set is a picture of precisely this dichotomy in the special case where 0 is used as the seed. If c=i, the orbit for $z^2 + i$ is given by $z_0 = 0$, $z_1 = i$, $z_2 = -1 + i$, $z_3 = -i$, $z_4 = -1 + i$, $z_5 = -i, \ldots$and we see that this orbit eventually cycles with period 2. If we change c to 2i, then the orbit behaves very differently $z_0 = 0$, $z_1 = 2i$, $z_2 = -4 + 2i$, $z_3 = 12 - 14i$ … and we see that this orbit tends to infinity in the complex plane.

Another method to created fractals is Newton's Method ([5], [13]), which is an iterative method that can be used to solve a variety of equations that can be differentiated. To be differentiable, the first derivative of the function must exist. Thus, for the equation $z^3 - 1 = 0$, the roots are $r_1 = 1$ and $r_{2,3} = \frac{-1 \pm i\sqrt{3}}{2}$. If these roots are plotted on graph paper with the OX axis being the real axis and the OY axis being the imaginary axis, the roots will lie equally spaced on a circle of radius 1. For any differentiable function f(z), Newton's Method is: $z_{n+1} = z_n - f(z_n)/f'(z_n)$, where $f'(z)$ is the derivative of f(z). With an initial guess close enough to a root of f(z), the sequence $z_0, z_1, z_2, z_3 \ldots$ will converge on the value of the root. This root is called the attractor for the sequence.

One of the simpler fractal shapes is the von Koch snowflake ([13]). The method of creating this shape is to repeatedly replace each line segment with the following line segments. The process starts with a single line segment and continues for ever. The first few iterations of this procedure are shown in figure 1.

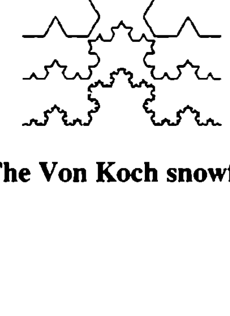

Figure 1: The Von Koch snowflake fractal

The Koch curve has an infinite length because each time the steps above are performed on each line segment of the figure there are four times as many line segments, the length of each being one-third the length of the segments in the previous stage. Hence the total length increases by one third and thus the length at step n will be $(4/3)^n$.

Starting with an equilateral triangle and applying the von Koch method, we can obtain the shape which is shown in fig. 2. Thus, the limit of an infinite construction that starts with a triangle and recursively replaces each line segment with a series of four line segments that form a triangular "bump". Each time new triangles are added; the perimeter of this shape grows by a factor of 4/3 and thus diverges to infinity with the number of iterations. The length of the Koch snowflake's boundary is therefore infinite, while its area remains finite.

Figure 2: Koch snowflake on triangle

In the same manner, if we start with a square and we apply the von Koch method, we can obtain the shape which is shown in fig. 3.

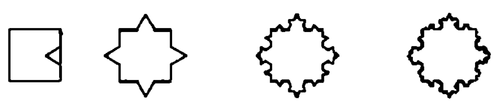

Figure 3: Koch snowflake on square

An interesting property of fractals is that they have non-integer dimensions. The dimension of a set is the number of independent parameters needed to describe a point in the set. Lines and curves are one-dimensional, planes and surfaces are two-dimensional, solids such as a cube are three dimensional, and so on. More formally, we say a set is n-dimensional if we need n independent variables to describe a neighborhood of any point. This notion of dimension is called the *topological dimension* of a set.

The **fractal dimension** ([9],[13]) is a statistical quantity that gives an indication of how completely a fractal appears to fill space. Fractal dimension can be calculated by taking the limit of the quotient of the log change in object size and the log change in measurement scale, as the measurement scale approaches zero. If we try to cover the unit square with little squares of side length s, we need $N(s)=\left(\frac{1}{s}\right)^2$ squares. For a segment of length 1, we need N(s)=1/s little squares. If we think of the square and segment as sitting in space and try to cover them with little cubes s on a side, we get the same answer. If we use the little cubes to cover a $1\times1\times1$ cube we need $N(s)=1/s^3$ cubes. Note that the exponent here is the same as the dimension of the thing we are trying to cover. Generally, if we denoted with D the dimension, we have $N(s)=\left(\frac{1}{s}\right)^D$. Closely related to this is the box-counting dimension, which considers, if the space were divided up into a grid of boxes of size ε, how the number of does boxes scale that would contain part of the attractor. One way to define the fractal dimension ([13]) is to ask, how many cubes of length s (or how many balls of radius s) it would take to cover the fractal. If we denoted with N(s) the number of boxes (cubs or balls) of length s required to cover the fractal, we obtain that the fractal dimension is given by:

$$D = \lim_{s \to 0} \frac{LN(N(s))}{LN\left(\frac{1}{s}\right)}$$

For example, consider a straight line. Now blow up the line by a factor of two. The line is now twice as long as before. D=LN 2 / LN 2 = 1, corresponding to dimension 1. Consider a square. Now blow up the square by a factor of two. The square is now 4 times as large as before (i.e. 4 original squares can be placed on the original square). D= LN 4 / LN 2 = 2, corresponding to dimension 2 for the square. Consider a snowflake (curve formed by repeatedly replacing ___ with _/_) where each of the 4 new lines is 1/3 the length of the old line. Blowing up the snowflake curve by a factor of 3 results in a snowflake curve 4 times as large (one of the old snowflake curves can be placed on each of the 4 segments _/_). Thus we have D= LN 4 / LN 3 = 1.261... Since the dimension 1.261 is larger than the dimension 1 of the lines making up the curve, the snowflake curve is a fractal.

Results and discussion

For example, broccoli is a fractal because it branches off into smaller and smaller pieces, which are similar in shape to the original. The fractal dimension of a broccoli is determined by calculating the slope of the graph of LN($N(s)$)/LN(s) (see [16]). Broccoli is especially interesting for its fractal properties. You can explore the concept of self-similarity by chopping broccoli.

Procedure

1. Measure and record the size of bunch, using either a micrometer (for smaller pieces) or a ruler (for larger pieces)
2. Break up the broccoli into smaller pieces and repeat step 1, three times in order to measure 3 levels of the fractal. We denoted by s = Size Range and by N(s) =Number of pieces.

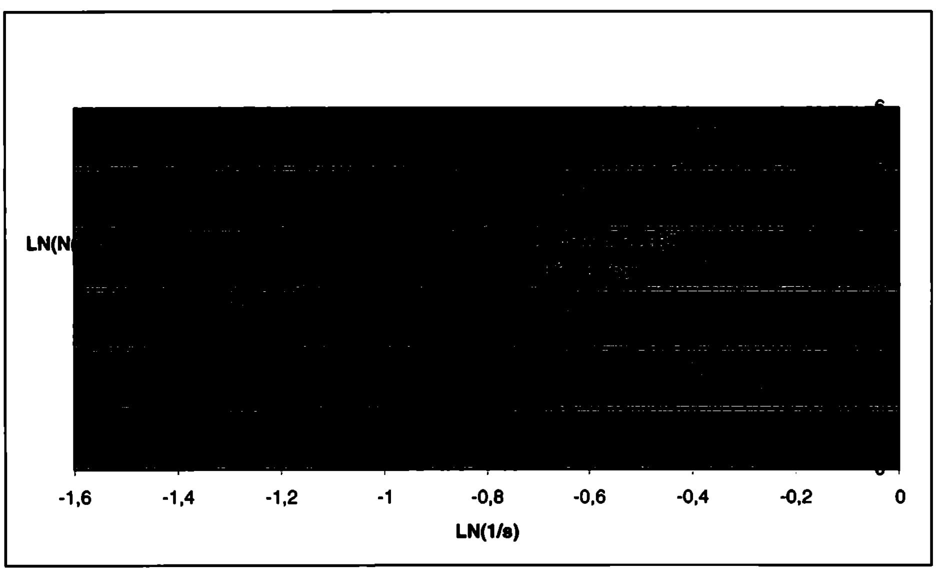

Figure 4: Fractal dimension of brocolli

The Fractal dimension of broccoli is the magnitude of the slope of the graph of LN(1/s) and LN(N(s)). The slope is 2.7463. Thus, the experimental fractal dimension of this studied broccoli is 2.7463.

Conclusions

A fractal does not have an integral dimension value. Fractals also describe many real-world objects, such as broccoli, ferns, clouds, mountains and coast lines that do not correspond to simple geometric shapes. Therefore, fractals have more and more applications in science. The main reason is that they very often describe the real world better than traditional mathematics and physics.

References

[1] Barnsley M., *Fractals Everywhere*, Academic press, 1988
[2] Becker, K.H., Dorfler M., *Dynamical Systems and Fractals*, Cambridge University Press, United Kingdom, 1983.
[3] Blumenthol L., Menger K., *Studies in Geometry*, W.H.Freeman & Co, 1970
[4] Ibrahim M., Krawczyk R. J., *Generating Fractals Based on Spatial Organizations*, (available on http://www.iit.edu/~krawczyk/fract01.pdf)
[5] Hutchinson J.E., *Fractals and Self Similarity*, Indiana University Mathematics Journal, Vol.30, No.5, pp.713-747, 1981.
[6] Mandelbrot B., *The Fractal Geometry of Nature*, W.H.Freeman & Co, NY, 1982
[7] Oppenheimer P.E., *Real Time Design and Animation of Fractal Plants and Trees*, Computer Graphics, 20, 4, 1986
[8] Peitgen H., Saupe D., *The Science of Fractal Images*, Springer-Verlag, NY, 1988
[9] Smith A.R., *Plants, Fractals and Formal Languages*, Computer Graphics, 18, 3, 1984,
[10] Shakiban C., Berstedt J.E., *Generalized Koch Snowflakes*, in Bridges: Mathematical Connections in Art, Music and Science, 1998
[11] http://home.att.net/%7EPaul.N.Lee/Fractal_Software.html#top
[12] http://archives.math.utk.edu/topics/fractals.html
[13] http://en.wikipedia.org/wiki/Fractal_dimension
[14] http://en.wikipedia.org/wiki/Koch_snowflake
[15] http://en.wikipedia.org/wiki/Mandelbrot_set
[16] http://hypertextbook.com/facts/2002/broccoli.shtml
[17] http://www.ultrafractal.com/