SOME PROPERTIES OF SUBMANIFOLDS OF THE RIEMANNIAN SPACE FORMS

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Rezumat

Teoria structurilor pe varietăţile Riemanniene este un subiect interesant şi modern al geometriei diferenţiale. În acest articol am introdus, mai întâi, noţiunile uzuale cum ar fi: curbura secţională, spaţiul de curbură constantă şi formulele de bază ale subvarietăţilor Riemanniene (formulele lui Gauss şi Weingarten, cât şi ecuaţiile lui Gauss şi Codazzi). În final voi da câteva exemple de subvarietăţi m-dimensionale ale spaţiului Euclidian R^n .

Résumé

La théorie des structures concernant les variétés Riemanniane est un sujet intéressant et moderne de la géométrie différentielle. Dans cet article, j'ai introduit, tout d'abord, des notions usuelles telles: la courbure sectionelle,l'éspace de courbure constante et les formules de basê les sous-variétés Riemanniene, ainsi que les équations de Gauss et Codazzi. Á la fin je donnerai quelques examples de sous-variétés m- dimensionalles de l'espace Euclidiene.

Abriss

Die Strukturentheorie für die riemannische Mannigfaltigkeiten ist einen sehr interessanter und modernem Subjekt für die Differentialgeometrie. In diesem Artikel führe ich ein zuerst die üblichen Begriffe wie, z.B.: die sektionelle Krümmung, der Konstantkrümmungsraum sowie auch die Basis Formeln den riemannischen Untermannigfaltigkeiten (die Gauss und Weingarten Formeln sowie auch die Gauss und Codazzi Gleichungen). Am Ende werde ich einige Beispiele den m - dimmensionellen Untermannigfaltigkeiten den euklidischen Raum R^n geben.

Introduction :

Let N be an n-dimensional Riemannian manifold with metric tensor field g, and R(X,Y) the Riemannian curvature transformation of the tangent space T_X (N) :

$$(1)\quad R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]},\quad (\forall)X,Y\in T_x(N)$$

The Riemannian curvature tensor field of type (0,4) of N, which denoted also R, is defined by:

$$(2)\quad R(X,Y,Z,T)=g(R(Z,T)Y,X)=g(R(X,Y)T,Z),\quad (\forall)X,Y,Z,T\in T_x(N)$$

If we take an orthonormal frame fields in N: E_1 , E_2 ,..., E_n , then

$$(3)\quad S(X,Y)=\sum_{i=1}^{n} g(R(E_i,X)Y,E_i)$$

defines a global trensor S of type (0,2), and it is called the Ricci tensor. From the tensor field S we define a global scalar field

$$(4)\quad r=\sum_{i=1}^{n} S(E_i,E_i)$$

and r is called scalar curvature of N. If n = 2, then G = ½ r is called the Gaussian curvature.

Let X and Y be two linearly independent vectors at a point x ∈ N, and p(X,Y) be the plane section spanned by X and Y. The sectional curvature k(p) for p is defined by:

$$(5)\quad k(p)=\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g^2(X,Y)}$$

If k(p)=c is a constant for all plane sections p in the tangent space $T_x(N)$, then N is called **the space of constant curvature**, and a complete simply connected Riemannian manifold of constant curvature is called **a space form** (we denote it by N^n (c)).

A Riemannian manifold of constant curvature is said to be **elliptic, hyperbolic or plat** (or locally Euclidian) according as the sectional curvature is positive, negative or zero.

Examples of space forms:

1. Let R^n be the affine space of dimension n with cartesian coordinates x^1, x^2, ..., x^n and let g be the Euclidian metric on R^n : $g = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2$. Then, R^n is a space form of zero curvature, and it is called the Euclidian n-space.

2. We denote by

$$N^n(c) = \{(x^1, x^2, \ldots, x^{n+1}) \in R^{n+1} / \sqrt{|k|}((x^1)^2 + \ldots + (x^n)^2) + \mathrm{sgn}(k)(x^{n+1})^2) - 2x^{n+1} = 0, x^{n+1} \geq 0\}$$

$$\textit{where} \quad \mathrm{sgn}(k) = \begin{cases} 1, k \geq 0 \\ -1, k < 0 \end{cases}$$

and $g=(dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2 + \mathrm{sgn}(k)(dx^{n+1})^2$ is the metric on R^{n+1}. Each $M^n(k)$ is complete, simply connected and of constant curvature c. Thus, $M^n(c)$ is a space form.

Let M be an m-dimensional manifold isometrically immersed in an n-dimensional Riemannian manifold N.

The second fundamental forms h and the Weingarten' operator A of submanifolds satisfy two compatibility equations calles the Gauss formula:

$$(6) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (\forall) X,Y \in T(M), \quad \nabla_X Y \in T(M), \quad h(X,Y) \in T^{\perp}(M)$$

and the Weingarten formula:

$$(7) \quad \overline{\nabla}_X V = -A_V X + D(X,V), \quad (\forall) X \in T(M), (\forall) V \in T^{\perp}(M), \quad A_V(X) \in T(M), \quad D(X,V) \in T^{\perp}(M)$$

A submanifolds M is said to be **totally geodesic** if its second fundamental form vanishes identicaly (that is h = 0 or echivalently A = 0).

If A_V is everywhere proportional to the identity transformation I, for a normal section V (that is A_V =fI, for some function f), then M is said to be **umbilical with respect to V**. If the submanifold M is umbilical with respect to every local normal section of M, then M is said to be **totally umbilical**.

The mean curvature vector H of M is defined to be

$$(8) \quad H = \frac{1}{m} Trace(h) = \frac{1}{m} \sum_{i=1}^{m} h(E_i, E_i) = \sum_{a=1}^{n-m} Trace(A_{\xi_a}),$$

where $\{E_1, E_2, \ldots, E_n\}$ is an orthonormal basis in T_x (M), and $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ is an orthonormal basis in $T(M)^{\perp}$.

If H = 0, then M is said to be **minimal**.

Remark: any submanifold M which is minimal and totally umbilical is geodesic.

The following equations hold for all vector fields X,Y,Z $\in$ T(M):

$$(9) \quad g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(h(X,W); g(Y,Z)) + g(h(Y,W); h(X,Z))$$

$$(10) \quad (\overline{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)$$

These are respectively the Gauss and Codazzi equations. Thus, (9) implies:

$$(11)\quad g(R(X,Y)Z;W) = c[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] +$$
$$+ g(h(Y,Z);h(X,W)) - g(h(X,Z);h(Y,W)) = c[g(Y,Z)g(X,W) -$$
$$- g(X,Z)g(Y,W)] + \sum_{a=1}^{n-m}[g(A_{\xi_a}Y,Z)g(A_{\xi_a}X,W) - g(A_{\xi_a}X,Z)g(A_{\xi_a}Y,W)]$$

If N is of constant curvature c, then equation of Gauss reduces to and $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ are an orthonormal frame fields normal to M (in $T(M)^{\perp}$).
Therefore, the Ricci tensor S of M is gived by:

$$(12)\quad S(X,Y) = (n-1)cg(X,Y) + \sum_a TraceA_{\xi_a} g(A_{\xi_a}X,Y) - \sum_a g(A_{\xi_a}X, A_{\xi_a}Y).$$

and the scalar curvature r of M is given by

$$(13)\quad r = n(n-1)c + \sum_{a=1}^{n-m}(TraceA_{\xi_a})^2 - \sum_{a=1}^{n-m}Trace(A_{\xi_a}{}^2)$$

In the case where M is a submanifold of N, and N is a space form of dimension n = m+1, if c = 1 $\Rightarrow$ N = S^{m+1} (the sphere), if c = 0 $\Rightarrow$ N = R^{m+1} (the Euclidian space), and if c = -1 $\Rightarrow$ N = H^{m+1} (the hiperbolic space).
Let A the shape operator of M associated to its unit normal ξ:

$$(14)\quad A_{\xi}X = AX = -\overline{\nabla}_X\xi$$

Thus, the Gauss and Codazzi ecuations become the folowing:

$$(15)\quad g(R(X,Y)Z;W) = c[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] +$$
$$+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W)$$
$$(16)\quad \nabla_X SY - \nabla_Y SX - S[X,Y] = 0.$$

Theorem1: A totally umbilical submanifold M of a Riemannian Manifold N (n $\geq$2) of constant curvature c is also of constant curvature.
Proof: Since M is totally umbilical, we have h(X,Y)=g(X,Y)·H, ($\forall$) X,Y $\in$ T(M). Thus, (11) implies:

$$(17)\quad g(R(X,Y)Z,W) = (c + |H|^2)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

This shows that M is of constant curvature $c+|H|^2$ for dimM=m > 2. If m=2, |H| is constant by equation of Codazzi.

In the next example we consider m-dimensional submanifold in an n-dimensional Euclidian space R^m with usual inner product <x,y>:

$$(18)\quad R^m = R^{p_1+1} \times \ldots \times R^{p_t+1}$$

where $p_1, p_2, \ldots, p_t > 0$, $p_1 + p_2 + \ldots + p_t = m$ and t = n – m.
We consider: $S^{p_i}(r_i) = \{x_i \in R^{p_i+1} / \langle x_i, x_i\rangle = r_i^2, i \in \{1,2,\ldots,t\}\}$.
Then, the pytagorean product:

$$\Pi S^{p_i}(r_i) = S^{p_1}(r_1) \times \ldots \times S^{p_t}(r_t) = \{(x_1,\ldots,x_t) \in R^n / x_i \in S^{p_i}(r_i), i \in \{1,\ldots,t\}\}$$

is an m-dimensional submanifold M^m of codimension n-m in R^n. The mean curvature vector H of M^m is given by

$$H=\frac{1}{m}\left(\frac{p_1}{r_1^2}x_1+\ldots+\frac{p_t}{r_t^2}x_t\right),\quad (x_1,\ldots,x_t)\in R^m, x_i\in S^{p_i}(r_i), i\in\{1,\ldots,t\}$$

$$\Rightarrow |H|^2=\frac{1}{m^2}\left(\frac{p_1^2}{r_1^4}x_1^2+\ldots+\frac{p_t^2}{r_t^4}x_t^2\right)=\frac{1}{m^2}\left(\frac{p_1^2}{r_1^2}+\ldots+\frac{p_t^2}{r_t^2}\right)$$

this shows that R^m is of constant curvature.

Theorem 2 ([4]): Let M be a m-dimensional complete submanifold of R^n with non-negative sectional curvature. Suppose that the normal connection of M is flat and the mean curvature vector of M is parallel. If the scalar curvature of M is constant, then M is a sphere $S^m(r)$, or a m-dimensional plan R^m, or a pytagorean product of the form:

$$(20)\quad S^{p_1}(r_1)\times\ldots\times S^{p_t}(r_t)\times R^p,\quad \sum_{i=1}^{t}p_i+p=m,\quad 1<t\le n-m\}$$

or a pytagorean product of the form:

$$(19)\quad S^{p_1}(r_1)\times\ldots\times S^{p_t}(r_t),\quad \sum_{i=1}^{t}p_i=m,\quad 1<t\le n-m\}$$

Theorem 3 ([4]): Let M be an n-dimensional compact submanifold of R^n with non-negative sectional curvature, and the normal connection of M is flat. If the mean curvature vector of M is parallel, then M is a sphere $S^m(r)$ or a pytagorean product of the form (19), which is of essential codimension t.

References:

1. Bang-Yen Chen : "Geometry of Submanifolds" – Marcel Dekker Inc , New York 1973
2. Gh. Gheorghiev, V. Oproiu – "Varietăţi diferenţiabile finit şi infinit dimensionale ", vol II, Editura Academiei RSR, 1979
3. S. Ianuş, Geometrie Diferenţială - Editura Academiei RSR, 1979
4. Kentaro Yano, Masahiro Kon: "Structures on manifolds" – Series in Pure Mathematics - vol 3- World Scientific Publishing Co
5. Kentaro Yano, S. Ishihara: "Submanifolds with parallel mean curvature vector" – Differential Geometry (1971)