THE REMARKABLE SPIRALS

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Rezumat

Spirala logaritmică are proprietatea că raza sa (distanţa de la polul O la punctul considerat al spiralei) creşte exponenţial cu unghiul dintre axa Ox şi raza corespunzătoare. Această curbă poartă denumirea şi de spirală echiunghiulară, deoarece unghiul dintre raza în punctul de coordonate polare (r,φ) şi tangenta la curbă în acest punct este constant). Spirala logaritmică poate fi construită cu ajutorul dreptunghiului de aur (ale cărei dimensiuni au raportul egal cu numărul de aur Φ). Spirala logaritmică este una dintre cele mai întâlnite curbe din natură . Astfel, ea se regăseşte în aranjamentul seminţelor de floarea soarelui, în forma unor scoici sau a cochiliilor de melc, fiind denumită "spira mirabillis") cât şi în arhitectura multor coloane antice greceşti.

Résumé

La spirale logarithmique (logistique) a la propriété rayon (la distance du pôle O au point considéré de la spirale) s'accroît de façon exponentielle avec l'angle entre l'axe Ox et le rayon considéré. Cette courbe s'appelle encore spirale equiangulaire, puisque l'angle entre le rayon dans le point de coordonate (r,φ) et la tangente à courbe dans ce point est constant. La spirale logarithmique (logistique) peut être construite à l'aide du rectangle d'or (dont les dimensions ont le rapport égal au numéro d'or Φ). La spirale logarithmique est l'une des courbes les plus recontrées dans la nature. Ainsi, on la rencontre dans la disposition des graines du tournesol, dans les certains coquillages, dans la coquille des escargots (étant nommée "spira mirabillis"), tout comme dans l'architecture de beaucoup de colones antiques grecques.

Abriss

Die logarithmische Spirale hat die Eigentchaft, dass ihr Radius (die Weite von dem Pol O bis zum zur Spirale gehörend betrachteten Punk) mit den Winkel zwischen der Achse Ox und dem betrachteten Radius Wächst. Diese Kurve heiβt auch gleichwinkligen Radius, weil der Winkel zwischen dem Radius im Punkt der Polarkoordinaten (r,φ) und der Tangente an der Kurve in diesem Punkt konstant ist. Die logarithmische Spirale kann mit der Hilfe des Goldrechteckes (dessen Gröβe das Verhältnis mit dem Goldnumer Φ gleich haben). Die logarithmische Spirale ist eine der teffenden Kurven in der Umwelt. So findet sie in der Stellung der Samen der Sonnenblume, in der Muschelform oder in der Form des Schneckenhauses, und sie wird als « spira mirrabilis » auch in der Architektur einiger antiken griechischen Säulen genannt.

Introduction

Equiangular spiral (also known as logarithmic spiral, Bernoulli spiral, the growth spiral, equiangular spiral, spira mirabilis, and logistique) describe a family of spirals which it is defined as a curve that cuts all radii vectors at a constant angle.

The spiral has been represented in nature for thousands of years in the shape of the nautilus shell, the arrangement of sunflower seeds, in flora and in fauna, among various other natural phenomena. The fundamental mathematical property of the equiangular (or logarithmic) spiral precisely corresponds to the biological principle that governs the growth of the mollusk's shell: the size increases but the shape is unaltered, it always remains similar to itself. This fact can be imitated by mathematical forms other than this spiral. A rectangle, a paralleogram, a cone, etc. can grow while remaining similar to itself in shape. This interested not only the Greeks of 500 B.C. but also the Egyptians a thousand years earlier.

Rene Descartes (1638) was the first to study the curve, but it was the favorite curve of Jakob Bernoulli (1654-1705). On his request his tombstone was decorated with a logarithmic spiral and accompanied by the following text: "eadem mutata resurgo" ("although changed, still remaining the same"). Therefore the curve is also called the Bernoulli spiral. Torricelli worked on the curve independently, and found the curve's length. The curve is also named by Fibonacci as the Fibonacci spiral.

Results and discussion

The logarithmic spiral can be constructed from equally spaced rays by starting at a point A along one ray, and drawing the perpendicular AB to a neighboring ray. As the number of rays approached infinity, the sequence of segments approaches the smooth logarithmic spiral.

For example, if the angle between two neighboring rays is 30^o and OA = a (fig. no. 1), then AB = a sin 30^0 , BC = a sin 30^o cos 30^o , CD = a sin 30^o (cos 30^o $)^2$, DE = a sin 30^o (cos 30^o $)^3$, ..., so, the length of segments AB, BC, CD,... are the terms of geometric series with ratio q = cos 30^0 < 1,

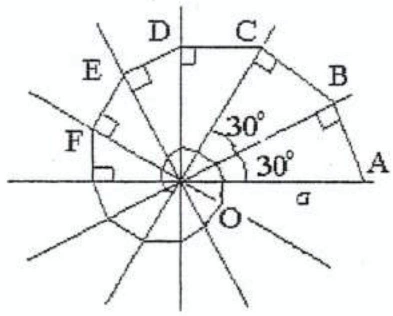

Fig. nr. 1: The construction of the spiral

and his sum is

$$S = a\sin 30^o \sum_{n=0}^{\infty} \cos^n 30^o = \frac{a\sin 30^0}{1-\cos 30^0} = a(2+\sqrt{3})$$

In the logarithmic spiral (fig. no. 2) the angle Ψ between the tangent and the radius (the polar tangent) is a constant.

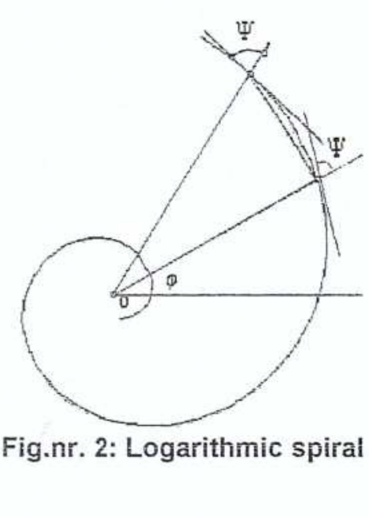

Fig.nr. 2: Logarithmic spiral

The polar equation of logarithmic spiral is given by

$$r = r(\varphi) = a \cdot e^{k \cdot \varphi}$$

where r is the distance from the origin, φ is the angle from Ox-axis and the radial line in the point of polar coordinates (r, ϕ), a and k are arbitrary constants.

It can be parametrically expressed as

$$x = x(\varphi) = r\cos\varphi, \quad y = y(\varphi) = r\sin\varphi$$

and

$$\cos\varphi = \frac{x}{r} = \frac{x}{\sqrt{r^2(\cos^2\varphi + \sin^2\varphi)}} = \frac{x}{\sqrt{x^2 + y^2}}$$

The rate of change of radius is:

$$\frac{dr}{d\varphi} = ake^{k\varphi} = kr \Rightarrow dr = kr \cdot d\varphi$$

and the angle between the tangent and radial line at the point of polar coordinate (r,φ) is

$$\psi = arctg\left(\frac{r}{dr/d\varphi}\right) = arctg\left(\frac{r}{kr}\right) = arctg(\frac{1}{k}) = arcctg(k), k > 0$$

So, as if k → 0 then ψ→ π/2, and the spiral approaches a circle.

The arc length of the logarithmic spiral can be expressed as :

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} d\varphi = \sqrt{r^2 + r^2k^2}\, d\varphi = r\sqrt{1 + k^2}\, \frac{dr}{r \cdot k}$$

$$s = \int_{\varphi_1}^{\varphi_2} \sqrt{x'^2 + y'^2}\, d\varphi = \int_{r_1}^{r_2} \frac{\sqrt{1 + k^2}}{k} dr = \frac{(r_2 - r_1)\sqrt{1 + k^2}}{k}$$

The curvature of the logarithmic spiral is:

$$K = \frac{x' y'' - y' x''}{\sqrt{\left(x'^2 + y'^2\right)^3}} = \frac{1}{a\sqrt{1 + k^2}\, e^{k\varphi}}$$

Another remarkable propertyes of this spiral is the relation between the golden mean and the equiangular spiral. So, an equiangular spiral can be derived from a golden rectangle. Many buildings, from Greek temples, are proportioned in accordance with the golden mean, a constant which is the ratio of the sides of a rectangle circumscribed about a logarithmic spiral.

The golden mean Φ (the positive solution of equation $x^2 - x - 1 = 0$) is defined such that *partitioning the original rectangle into a square and a new rectangle (fig. no. 3)*

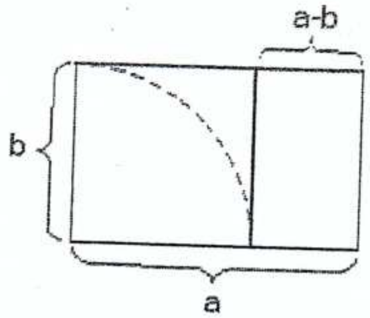

Fig. nr. 3: Golden rectangle.

results a new rectangle having sides into the same ratio Φ:

$$\frac{a}{b} = \frac{b}{a-b} \Rightarrow a^2 - ab - b^2 = 0 \Rightarrow \left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0 \Rightarrow \frac{a}{b} = \frac{1+\sqrt{5}}{2} = \phi$$

In the figure no.4 is a set of golden rectangles which inscribe a logaritmic spiral.

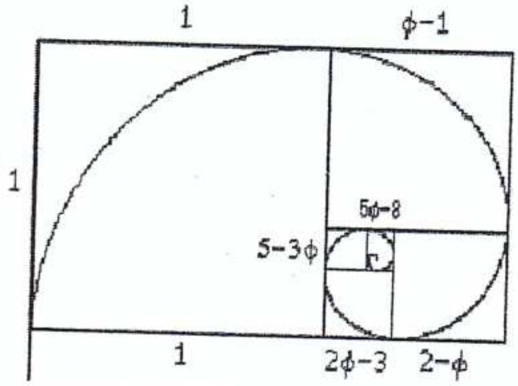

Fig. nr. 4: Golden rectangles which inscribe a logarithmic spiral

Starting with a single golden rectangle (of initial length Φ and width 1), there is a natural sequence of nested golden rectangles obtained by removing the leftmost square from the first rectangle, the topmost square from the second rectangle, etc. The length and width of the n-th golden rectangle can be written as linear expressions a+bΦ, where the coefficients a and b are always Fibonacci numbers: 1, 2, 3, 5, 8,…., $a_{n+2} = a_{n+1} + a_n$.

In **conclusion**, the remarkable qualities of the logarithmic spiral are the following:

- In the logarithmic spiral the angle between the tangent and the radius (the polar tangent) is a constant. This property gives the spiral the name of **equiangular spiral**.
- If the angle between the tangent and the radius is $\pi/2$ the result is a **circle**.

- The distances where a radius from the origin meets the curve are in geometric progression.
- The radius grows exponentially with the angle. The logarithmic relation between radius and angle leads to the name of **logarithmic spiral**.

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